Single-mode nonclassicality criteria via Holstein-Primakoff transformation

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Recently, two quantifications for nonclassicality of a single-mode field are shown to be equivalent; (i) the rank of entanglement it can generate by a beam-splitter and (ii) the number of terms needed to expand it as superposition of coherent states. We show that nonclassicality criteria can be obtained with an alternative approach. The rank of two-mode entanglement among 2-level identical particles converges to the rank of single-mode nonclassicality within the Holstein-Primakoff transformation, at the large particle number limit. In particular, we show that the entanglement criterion of Hillery & Zubairy converges to the Mandel's Q-parameter which is used to reveal nonclassicality, and spin-squeezing criterion of Sørensen *et al.* converges to single-mode squeezing condition. We obtain additional nonclassicality criteria not existing in the literature. We also discuss if single-mode nonclassicality can be visualized as the entanglement of space generating the photons. Moreover, in a forthcoming study we show that, linear optical response of an optomechanical cavity becomes noncausal above the critical coupling where output single-mode field becomes nonclassical.

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I. INTRODUCTION

In almost all cases, examining the correlations between two pulses, one can distinguish if they are emitted by two independent sources or emitted by a single source but divided by a beam splitter (BS). However, the two situations cannot be distinguished [1] if the sources radiate the coherent states of Glauber [2]. For this reason, coherent states are referred as classical, where exotic quantum features disappear. Nonclassicality is shown to be a necessary condition for obtaining entangled modes through a BS [3, 4].

Single-mode (SM) nonclassicality is a desirable quantum feature which is needed for quantum teleportation [5–7], as generator of mode entanglement [8, 9], in producing single-photon pulses (anti-bunching) [10–12], for transferring squeezing to spin ensembles [13], and for measurements below the standard quantum limit (SQL) [14–17]. Quantifying the measure of single-mode nonclassicality is not an apparent issue, unlike two-mode entanglement [18–21], even for pure states. Defining nonclassicality by the negativity Wigner function [22, 23] or Sudarshan-Glauber P distribution [24, 25] is inadequate since these functions are positive for pure squeezed states [26, 27] even though they are nonclassical.

A more appropriate definition and measure for single-mode nonclassicality has appeared in in 2005 in terms of its ability to produce two-mode entanglement [28, 29]. The maximum possible entanglement generated by a single-mode field at the two output modes of a BS is referred as the entanglement potential. Such a classification of nonclassicality is shown [30] to be in parallel with the Mandel's Q-parameter condition [31].

Recently, Vogel and Sperling [32, 33] derived an intriguing connection between the number of coherent states (in the superposition) required to compose a nonclassical single-mode

$$|\psi_{\rm Ncl}\rangle = \sum_{i=1}^{r} \kappa_i |\alpha_i\rangle \tag{1}$$

and the rank of the two-mode Schmidt decomposition this field generates through a BS,

$$|\psi_{\rm Ent}\rangle = \sum_{i=1}^{r} \lambda_i |a_i\rangle \otimes |b_i\rangle \ . \tag{2}$$

In other words, superposition of r coherent states yields a superposition of an output state (among modes a and b) with Schmidt rank r [32, 33], which determines the tank of two-mode entanglement.

States of N identical particles also exhibit some analogies with Fock number and coherent states [34–36]. Identical particles can occupy only the symmetric Dicke states [37] where particles have exchange symmetry regarding internal states [38–40]. A Dicke state transforms to a Fock number state [34] within the large number of particles limit. Furthermore, operations in Dicke states can be mapped to single-mode states by Holstein-Primakoff transformation [41, 42], i.e. $\hat{S}_+ \to \hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}} \to \sqrt{N} \hat{b}^\dagger$ where $\hat{S_+}$ is the ladder operator for collective Dicke (spin) states and $\hat{b}$ is annihilation operator for a single-mode field. Dicke states are entangled, i.e. density matric cannot be written as [43, 45, 68]

$$\hat{\rho} = \sum_k P_k \ \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes \ldots \otimes \hat{\rho}_k^{(N)} \ , \tag{3}$$

where $\hat{\rho}_k^{(i)}$ is the density matrix belonging to the $i^{\rm th}$ particle.

On the other hand, in difference to Dicke number states, atomic coherent states (ACS) [36] have a distinct

feature that they can be written as multiplies of of single-particle states

$$|\psi_{\rm ACS}(z)\rangle \sim \big(|g\rangle_1 + z|e\rangle_1\big) \otimes \ldots \otimes \big(|g\rangle_N + z|e\rangle_N\big), \quad (4)$$

where $|g\rangle_i$ ($|e\rangle_i$) is the ground (excited) state of the $i^{\rm th}$ particle. Similar to Dicke states, ACS states transform to single-mode coherent states [34, 35] in the large particle number limit.

In this paper, we argue the analogy between a non-classical single-mode state and inseparability of N 2-level identical particles [43, 45, 68]. We illustrate that Schmidt ranks of N particle inseparable state and single-mode coherent state [32, 33] are equivalent under the Holstein-Primakoff transformation. We demonstrate that this method is also equivalent to determination of entanglement potential [28, 29] via beam splitter.

In particular, we derive the Mandel's Q-parameter condition [31] from Hillert & Zubairy entanglement criterion [20], both using the beam-splitter approach [28] and Holstein-Primakoff transformation. We show that they are equivalent. Using Hillery & Zubairy criteria we derive Mandel's nonclassicality criteria for higher order correlation factors, e.g. $g^{(3)}$ and $g^{(4)}$. We also demonstrate that N-particle spin-squeezing criterion of Sørensen *et al.* [43] transforms to single-mode squeezing condition within the Holstein-Primakoff transformation.

Nonclassical single-mode states not only can generate multimode entangled photons through a BS, but they can also transfer entanglement or squeezing to atomic ensembles [13] with linear interactions [52]. This is similar to transfer of squeezing from two-mode entangled Stokes photons to atomic ensembles [50] or transfer of entanglement between two interacting ensembles [51]. However, one naturally avoids referring the single-mode states as entangled, since he/she cannot find two or more parts to associate for defining the inseparability. In the picture, presented in this paper, single-mode nonclassicality attains an apparent physical meaning that is the entanglement of the vacuum generating the photons, which can lead to uncontrolled [46] superluminal communication within the extent of the pulse.

The paper is organized as follows. In Sec. II A, we review the collective excitations of a N-particle system. We introduce Dick number and atomic coherent states which approach to Fock number and coherent single-mode states, respectively, for large number of particles. We introduce the connections between the mode operators and the Holstein-Primakoff transformation. In Sec. II B, we reveal the equivalence between the rank of N-particle inseparability and the number of the terms needed in the coherent state expansion [32]. We show that two-mode entanglement of particles is closely related to two-mode entanglement a single-mode field generates at the BS output. In Sec. III A, we derive the Mandel's Q-parameter condition using Holstein-Primakoff transformation. In Sec. III B, we derive the same condition alternatively using the BS approach. In Sec. III C, we show that spins-squeezing criterion of Sørensen *et al.* [43] converges to single-mode squeezing condition within Holstein-Primakoff transformation. However, the same condition cannot be derived using BS approach. Sec.IV includes our conclusions.

II. EQUIVALENCE OF NONCLASSICALITY TO N-PARTICLE INSEPARABILITY

A. Atomic and single-mode coherent states

States of N 2-level particles can be represented by angular momentum addition theorem of N spin-1/2 particles [36] called as Dicke states, see figure 1 in Ref. [37]. Dicke states are usually used to describe collective phenomena in ensembles such as superradiance [39, 41]. Regarding identical particles, bosons [47] and fermions [40], only the symmetric set (maximum cooperation $r = N/2$) of Dicke states can be occupied [38, 56]. An atomic coherent state (ACS) expended in Eq. (4), $|S, z\rangle and S = N/2$, is generated from the ground state

$$|S, -S\rangle = |g\rangle_1 \otimes |g\rangle_2 \ldots \otimes |g\rangle_N \quad (5)$$

(all particles are in ground state) by applying the collective atomic displacement operator

$$\hat{D}_a = e^{\xi \hat{S}_+ - \xi^* \hat{S}_-} \quad \text{as} \quad (6)$$

$$\begin{aligned} |S, z\rangle &= \hat{D}_a |S, -S\rangle \\ &= \frac{1}{|1+|z|^2|^{N/2}} \sum_{m=-S}^{S} \binom{N}{S+m}^{1/2} z^{S+m} |S, m\rangle \quad (7) \end{aligned}$$

where $\hat{S}_\pm$ are ladder operators with

$$\hat{S}_\pm |S, m\rangle \left[(S \mp m)(S \pm 1)\right]^{1/2} |S, m \pm 1\rangle \quad (8)$$

and $z = \tan|\xi| e^{i \arg\{\xi\}}$.

An atomic coherent state has the special property that it is separable as the multiplication of single atom states (in parallel with Eq. 5)

$$|S, z\rangle = c\big(|g\rangle_1 + z|e\rangle_1\big) \otimes \ldots \otimes \big(|g\rangle_N + z|e\rangle_N\big), \quad (9)$$

where c stands for the normalization constant $1/(1 + |z|^2)^{1/2}$. Hence, ACS is not entangled in the sense of definition of Sørensen *et al.* [43, 68]

$$\hat{\rho} = \sum_k P_k \, \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes \ldots \otimes \hat{\rho}_k^{(N)} \quad . \quad (10)$$

Since the particles are identical, their algebra can be carried out using two-mode annihilation operators $\hat{c}_g$ and $\hat{c}_e$, see Appendix 1A in Ref. [48]. These two operators are related to collective angular momentum operators by

$$\hat{S}_+ = \hat{c}_e^\dagger \hat{c}_g, \quad \hat{S}_- = \hat{c}_g^\dagger \hat{c}_e \quad \text{and} \quad \hat{S}_z = (\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g)/2 \quad (11)$$

Symmetric Dicke (number) states are also number states for $\hat{c}_g$ and $\hat{c}_e$ operators.

A parallel entanglement definition [19] can be made as

$$\hat{\rho} = \sum_i \mathrm{P}_i \rho_{i1} \otimes \rho_{i2} \tag{12}$$

in the sense of mode-inseparability of the indistinguishable particles, compared to the notion of multiplies of single-particle states [43] as given in Eq. (10). Duality is lifted by Dalton *et al.* [45] where it is shown that inseparability in terms of single-particle states [43, 68] already necessitates the two-mode entanglement [19]. In fact, this equivalence is also indirectly shown by Voget and Sperling [32, 33], which becomes apparent after Eq. (19), below.

In the large number of particles limit, $N \to \infty$, ACSs transform to coherent states of photons [34, 35]. Similarly, symmetric Dicke states correspond to Fock number states of a single-mode radiation. A transformation, which has the same notion, can be carried out using Holstein-Primakoff transformation [41, 42]

$$\begin{aligned} &\hat{S}_+ = \hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}} \quad , \quad \hat{S}_- = \sqrt{N - \hat{b}^\dagger \hat{b}}\, \hat{b} \\ &\text{and} \quad \hat{S}_z = \hat{b}^\dagger \hat{b} - N/2 \end{aligned} \tag{13}$$

without referring to states explicitly. When N is sufficiently large, ladder operators transform to single-mode annihilation/creation operators

$$\hat{S}_+ \to \sqrt{N}\hat{b}^\dagger \quad , \quad \hat{S}_- \to \sqrt{N}\hat{b} \quad \text{and} \quad \hat{S}_z \to -N/2 + \hat{b}^\dagger \hat{b} \, , \tag{14}$$

where $\hat{b}$ corresponds to $\hat{c}_e$ operator in two-mode representation in Eq. 11.

It is worth noting that collective atomic displacement operator $\hat{D}_a(z)$, in Eq. (6), transforms to single-mode displacement operator

$$\hat{D} = e^{\alpha \hat{b}^\dagger - \alpha^* \hat{b}} \tag{15}$$

which leaves noise elements (or fluctuations) unaffected.

We remind that one expects nonclassicality criteria to be dependent on the noise spectrum of the single-mode field [28], since entanglement criteria depend on covariance matrices [18, 19, 21, 66].

B. Equivalence of Schmidt Ranks

Any N-particle state (for indistinguishable particles) can be written as superposition of ACSs,

$$|\psi_N\rangle = \sum_{i=1}^{r} \kappa_i \; |N/2, z_i\rangle \; , \tag{16}$$

or in superposition of separable N-particle states [43]

$$|\psi_N\rangle = \sum_{i=1}^{r} \kappa_i \; (|g\rangle_1 + z_i |e\rangle_1) \otimes \ldots \otimes (|g\rangle_N + z_i |e\rangle_N) \, . \tag{17}$$

Hence, the rank of the N-particle entanglement is r and converges to the degree of nonclassicality [32, 33]

$$|\psi\rangle = \sum_{i=1}^{r} \kappa_i \; |\alpha_i\rangle \tag{18}$$

of the single-mode field.

Since N-particle entanglement [43] also implies two-mode entanglement [19], as it is stated in Ref. [45], evolution under the beam splitter hamiltonian

$$e^{\xi \hat{a}_2^\dagger \hat{a}_1 - \xi^* \hat{a}_1^\dagger \hat{a}_2} \equiv e^{\xi \hat{a}_2^\dagger \hat{b} - \xi^* \hat{b}^\dagger \hat{a}_2} \equiv e^{\xi \hat{a}_2^\dagger \hat{c}_e - \xi^* \hat{c}_e^\dagger \hat{a}_2} \tag{19}$$

maps the two-mode entanglement ($\hat{c}_g - \hat{c}_e$) of N-particles to two-mode entanglement of BS output modes ($\hat{a}_1 - \hat{a}_2$). We changed $\hat{a}_1$ to $\hat{c}_e$ (or to $\hat{b}$), since it essentially acts on the single-mode input in BS, which represents excitation over an N-particle system.

Two-mode entanglement in the BS output [28] and the ranking the number of coherent states required in superposition are shown [32], in Eq. (18), to be equivalent descriptions of the nonclassicality. Above, we also show that N-particle entanglement and ranking of coherent states are also equivalent. Hence, it follows that, the ranks of the N-particle entanglement and two-mode entanglement at BS are also equivalent.

The process taking place in a BS can also be visualized as follows. The physics of the input $\hat{b}$ mode is equivalent to excitations of N-particle system. The two-mode entanglement of N-identical particles is transferred to in between the two output modes of the BS, in the same manner as in Ref. [13].

III. NONCLASSICALITY CONDITIONS VIA HOLSTEIN-PRIMAKOFF TRANSFORMATION

We demonstrated the the equivalence of three criteria (i) the rank of N-particle inseparability, (ii) number of elements needed in superposition of a single-mode mode field, and by Ref. [32], (iii) two-mode entanglement generated at the BS output from this nonclassical field.

Next, we show that two-mode entanglement criteria (or equivalently N-particle entanglement [45]) can be practically converted to single-mode nonclassicality criteria using the Holstein-Primakoff transformation [41, 42]. In particular, we show that Hillery & Zubairy criterion [20] converges to Mandel's Q-parameter [31], and spin-squeezing criterion of Sørensen *et al.* [43] for N-particle entanglement transforms to squeezing condition for single-mode field.

In addition, we show that Mandel's Q-parameter condition can equivalently be derived using beam-splitter formalism by a lower order Hillery & Zubairy (H&Z) criterion. On the other hand, single-mode squeezing condition cannot be derived from the spin-squeezing criterion [43]. We also derive conditions, familiar to Mandel's Q-parameter condition, for higher order correlations $g^{(3)}$

and $g^{(4)}$ [48], which need not necessarily be equivalent to the one for $g^{(2)}$.

A. Mandel's Q-parameter from H&Z criterion

A set of sufficient criteria for two-mode entanglement is given by Hillery & Zubairy (H&Z) [20] as

$$\left|\langle \hat{c}_g^m (\hat{c}_e^\dagger)^n \rangle\right|^2 > \left\langle (\hat{c}_g^\dagger)^m (\hat{c}_g)^m (\hat{c}_e^\dagger)^n (\hat{c}_e)^n \right\rangle \tag{20}$$

which implies inseparability for any n,m=1,2... values. $\hat{c}_g$ and $\hat{c}_e$ are annihilation operators for the two inseparable modes. In order to obtain the Mandel's Q-parameter, we consider the case with $m = n = 2$ as

$$\left|\langle \hat{c}_g^2 (\hat{c}_e^\dagger)^2 \rangle\right|^2 > \left\langle (\hat{c}_g^\dagger)^2 (\hat{c}_g)^2 (\hat{c}_e^\dagger)^2 (\hat{c}_e)^2 \right\rangle \tag{21}$$

We aim to write Eq. (21) in terms of collective spin operators, given in Eq. 11. The term on the right hand side (RHS) of Eq.. (21) can be transformed to

$$(\hat{c}_g^\dagger)^2 \hat{c}_g^2 (\hat{c}_e^\dagger)^2 \hat{c}_e^2 = \hat{c}_g^2 (\hat{c}_g^\dagger)^2 (\hat{c}_e^\dagger)^2 \hat{c}_e^2 - 4\hat{c}_g \hat{c}_g^\dagger (\hat{c}_e^\dagger)^2 \hat{c}_e^2 + 2(\hat{c}_e^\dagger)^2 \hat{c}_e^2 \tag{22}$$

using the commutation relations. The first term on the RHS of Eq. (22) is $\hat{S}_+^2 \hat{S}_-^2$. We note that for an ensemble with N is very large compared to the excitation $\langle \hat{c}_e^\dagger \hat{c} \rangle$, the first term on the RHS is proportional to N^2 whereas the second and third terms are proportional to N and 1, respectively. Hence, last two terms can b en neglected for $N \to \infty$. Therefore, Eq. (21) takes the form

$$\left|\langle \hat{c}_g^2 (\hat{c}_e^\dagger)^2 \rangle\right|^2 > \langle (\hat{c}_e^\dagger)^2 \hat{c}_g^2 \, \hat{c}_e^2 (\hat{c}_g^\dagger)^2 \rangle = \langle \hat{S}_+^2 \hat{S}_-^2 \rangle \, . \tag{23}$$

The $\hat{S}_+^2 \hat{S}_-^2$ term will transform to the desired form, $(\hat{b}^\dagger)^2 \hat{b}^2$, after Holstein-Primakoff transformation. The the term on the left hand side (LHS) can be related with the number operator $\langle \hat{b}^\dagger \hat{b} \rangle$ using the Cauchy-Schwartz inequality

$$\langle \hat{c}_g^\dagger \hat{c}_e \, \hat{c}_e^\dagger \hat{c}_g \rangle \langle \hat{c}_g \hat{c}_e^\dagger \, \hat{c}_g^\dagger \hat{c}_e \rangle \geq \langle \hat{c}_g^2 (\hat{c}_e^\dagger)^2 \rangle \tag{24}$$

where LHS can be written in terms of collective spin operators as $\langle \hat{S}_- \hat{S}_+ \rangle \langle \hat{S}_- \hat{S}_+ \rangle$ which converges to $\langle \hat{S}_+ \hat{S}_- \rangle^2$ neglecting the term $\sim N$ compared to N^2 term. Hence, Eq. (21) becomes

$$|\langle \hat{S}_+ \hat{S}_- \rangle|^2 > \langle \hat{S}_+^2 \hat{S}_-^2 \rangle \langle \hat{S}_+^2 \hat{S}_-^2 \rangle \tag{25}$$

for N is sufficiently large. Applying the Holstein-Primakoff transformation, given in Eq. (13), inequality (25) becomes

$$|\langle \hat{b}^\dagger \hat{b} \rangle|^2 > \langle (\hat{b}^\dagger)^2 \hat{b}^2 \rangle \tag{26}$$

that is the Mandel's Q-parameter [31] for a single-mode field.

Using the inequality set (20), one can identify higher order nonclassicality conditions

$$|\langle \hat{b}^\dagger \hat{b} \rangle|^\ell > \langle (\hat{b}^\dagger)^\ell \hat{b}^\ell \rangle \, , \tag{27}$$

other than the standard form Eq. (26), for Q-parameter with ℓ integer. In addition to set of inequality (27), one also obtains

$$|\langle \hat{b} \rangle|^2 > \langle \hat{b}^\dagger \hat{b} \rangle \tag{28}$$

for $m = n = 1$ in Eq. (20).

B. Mandel's Q-parameter using BS approach

One may reach Mandel's Q-parameter condition for a single-mode field, alternatively by examining the entanglement of the two output modes of the BS [28, 32, 33], $\hat{a}_1$ and $\hat{a}_2$. Output modes can be determined [1, 3, 49] using the beam splitter operator

$$\hat{B}(\xi) = e^{\xi \hat{a}_2^\dagger \hat{a}_1 - \xi^* \hat{a}_1^\dagger \hat{a}_2} \, . \tag{29}$$

One acts $\hat{B}(\xi)$ on the initial separable state of two modes,

$$|\psi_{12}\rangle = |\psi_a\rangle_1 \otimes |0\rangle_2 \tag{30}$$

where $|\psi_a\rangle$ is the state of the single-mode field ($\hat{a}$) incident to the BS, $|\psi_a\rangle = \sum_{n=0}^\infty c_n |n\rangle$, placed into the initial state of the first mode ($\hat{a}_1$). This method is equivalent to $|\psi_{12}\rangle = f(\mu_1 \hat{a}_1^\dagger + \mu_2 \hat{a}_2^\dagger)|0\rangle_1 \otimes |0\rangle_2$ transform [1] on the two-mode wave function, where function f is defined due to single-mode wave function $|\psi_a\rangle = \left(\sum_{n=0}^\infty d_n (\hat{a}^\dagger)^n\right)|0\rangle_a$ as $f(z) = \sum_{n=0}^\infty d_n z^n$. Coefficients $\mu_1 = te^{i\phi}$ and $\mu_2 = r$ follow from the BS transformation [3, 49]

$$\hat{a}_1(\xi) = \hat{B}^\dagger(\xi) \hat{a}_1 \hat{B}(\xi) = te^{i\phi} \hat{a}_1 + r\hat{a}_2 \, , \tag{31a}$$

$$\hat{a}_2(\xi) = \hat{B}^\dagger(\xi) \hat{a}_2 \hat{B}(\xi) = -r\hat{a}_1 + te^{-i\phi} \hat{a}_2 \, , \tag{31b}$$

where t^2, r^2 are transmission, reflection coefficients and ϕ is the phase of the BS.

An expectation value including $\hat{a}_1$, $\hat{a}_2$ operators, for example

$$\langle \hat{a}_1 \hat{a}_2^\dagger \rangle = {}_2\langle 0| \otimes {}_1\langle \psi_a | \hat{B}^\dagger(\xi) \hat{a}_1 \hat{a}_2^\dagger \hat{B}(\xi) |\psi_a\rangle_1 \otimes |0\rangle_2 \, , \tag{32}$$

can be evaluated performing the transformations (31a) and (31b).

For the purpose of obtaining the Mandel's Q-parameter condition via BS approach, e use two-mode entanglement criterion of H&Z, given in inequality set (20), with a lower degree compared to the one used in Holstein-Primakoff transformation (21),

$$|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|^2 > \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle \, . \tag{33}$$

Using the transformations (31a) and (31b) one can relate terms in Eq. (33) to the ones for single-mode field ($\hat{a}$) as

$$\langle \hat{a}_1 \hat{a}_2^\dagger \rangle = -rte^{i\phi} \langle \hat{a}^\dagger \hat{a} \rangle \tag{34}$$

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle = t^2 r^2 \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle \tag{35}$$

which simply gives the condition for Mandel's Q-parameter

$$|\langle \hat{a}^\dagger \hat{a} \rangle|^2 > \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle \ . \tag{36}$$

We note that equivalence between the result with Holstein-Primakoff transformation, given in Eq. (26), and with BS approach (two-mode), given in Eq. (36), originates from the equivalence of N-particle entanglement to the two-mode entanglement [45], as mentioned at the end of Sec. II B.

C. Single-mode squeezing from spin-squeezing criterion

Sørensen *et al.* [43] derived an inseparability criterion witnessing the entanglement of N particles

$$\xi^2 \equiv \frac{N(\Delta \hat{S}_{\mathbf{n_1}})}{\langle \hat{S}_{\mathbf{n_2}} \rangle + \langle \hat{S}_{\mathbf{n_3}} \rangle} < 1 \tag{37}$$

by generalizing their method for bipartite entanglement [19]. In Eq. (37), $\hat{S_{\mathbf{n}}}$ are the collective spin operators for N-particle ensemble, see Eq. (8).

By performing the Holstein-Primakoff transformation, see Eq. (13), and taking the limit as $N \to \infty$ as in Sec. III A, one can easily obtain

$$(\Delta \hat{x}_b)^2 < 1/2 \tag{38}$$

that is the squeezing condition for the single-mode field $\hat{b}$, with $\hat{x}_b = (\hat{b}^\dagger + \hat{b})/\sqrt{2}$.

Unfortunately, one cannot deduce the squeezing condition (38) using the BS formalism (at least we could not manage to do it) analytically. In the case of H&Z criterion, the coefficients r, t and $e^{i\phi}$ could be cancelled in Sec. III A. However, in this case numerical minimization with respect to r and ϕ is required [28].

IV. DISCUSSIONS AND CONCLUSIONS

The symmetric set of Dicke states (see figure 1 in Ref. [37]) for a system of N identical 2-level particles can be mapped onto the quantum states of a single-mode field. In particular, Dicke number states and atomic coherent states converge to single-mode Fock number states and coherent states, respectively, at the large particle limit.

Recently Vogel and Sperling demonstrated [32, 33] that (i) the number of terms in the coherent state expansion of a single-mode state (nonclassicality) is equal to the (ii) rank of two-mode entanglement this single-mode state generates at the output of a beam-splitter (BS). We additionally show that, (i) number of terms in coherent state expansion of the single-mode is equal to the (iii) rank of N-particle entanglement [43, 68]. The single-mode is the quasiparticle excitation of N identical 2-level particles. Hence, the three quantities (i, ii, and iii) become equivalent to each other.

We utilize the Holstein-Primakoff transformation [41, 42] as a tool for nonclassicality. We convert the criteria for N-particle inseparability [43] and two-mode entanglement [20] (which is already a prerequisite for N-particle entanglement [45]) into conditions for single-mode nonclassicality.

We show that Mandel's Q-parameter condition, for single-mode nonclassicality, can be obtained from Hillery & Zubairy criteria [20], both using Holstein-Primakoff transformation and beam-splitter approaches. On the other hand, situation is different for single-mode squeezing criterion. One can easily obtain the analytical form for squeezing criterion using Holstein-Primakoff transformation. However, numerical maximization is required for the beam-splitter approach. Holstein-Primakoff transformation also works for mixed single-mode states, since two-mode entanglement criteria [20, 43] were deduced already for mixed quantum states.

Unfortunately, two entanglement criteria [18, 19] – which are both necessary and sufficient criteria for Gaussian states– require the evaluation of non-number conserving terms like $\langle \hat{c}_g \hat{c}_e \rangle$. Hence, this kind of criteria cannot be trasnformed to single-mode nonclassicality conditions using the Holstein-Primakoff approach. For this reason, in Ref. [65] we use the beam-splitter approach to deduce the degree of single-mode nonclassicality from Simon-Peres-Horodecki criterion [18, 66, 67]. One expects a further connection between the entanglement depth [68] of N-particle inseparability and the degree of single-mode nonclassicality (of quasiparticles) due to Holstein-Primakoff transformation.

Several authors [58–60] showed the equivalence of entanglement to a wormhole connecting the two inseparable particles. In this paper, we show that nonclassicality of quasiparticle excitations of an N-particle system corresponds to collective entanglement among these particles. Following analogy can be considered (only) for visualization purposes. Nonclassical states of a phonon (sound) field [70] corresponds to the entanglement of vibrating atoms within the extent of the phononic field. In this picture, nonclassicality of electromagnetic radiation corresponds to entanglement of the different parts of the vacuum generating photons as quasiparticles [72, 73]. Therefore, different parts of the space (e.g. source and potential) may perform uncontrolled [46] superluminal communication. In field theory [71] and in electromagnetics {see Eq.s (6.42) and (6.45) in Ref. [61] and Sec. IV in Ref. [69]}, superluminal communication may show itself by noncausal behaviour of fields.

In a forthcoming study [69], we demonstrate the take place of such a phenomenon. The output field of an optomechanical system becomes nonclassical above a critical cavity-mirror coupling strength $g_c = 2\sqrt{\gamma_m/\gamma_c}\omega_m$ [69]. Here, γ_m (γ_c) is the mechanical (cavity) damping rate and ω_m is the resonance of mechanical oscillator.

If one has no information about the setup in the cavity, he/she may assign a refractive index $n(\omega)$ to the cavity slab according to the measured reflected and transmitted pulses. We show that this index $n(\omega)$ becomes noncausal (pole moves to the upper-half of the complex-ω plane) at exactly the same critical strength for cavity-mirror coupling ($g = g_c$). Moreover, such a behavior emerges due to the noncausal matching of incident and reflected waves in the same medium, not due to boundary conditions. As a final note, the superluminal velocity mentioned here is completely different than the superluminal group velocity in a casual medium [62–64].

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